International Journal of Applied Sciences: Current and Future Research Trends
(IJASCFRT)
ISSN (Print) , ISSN (Online)

https://ijascfrtjournal.isrra.org/index.php/Applied_Sciences_Journal

Stationary Distribution of a Mathematical Model

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Abstract

This paper aims to study a new class of a stochastic SIS (susceptible, infected, susceptible) epidemic model where the transmission coefficient of the disease and the death rate are perturbed. By using the Khasminski theory, we establish suitable conditions under which the stochastic SIS model has a unique stationary distribution. The stationary of such model means that the disease will prevail.

Keywords: Stochastic differential equation ; Itô formula ; Stationary distribution ; Ergodicity.

1. Introduction

Infectious diseases killed more beings humans in the course of history than any other cause and caused heavy economic losses as well. The mathematical modeling of the infectious disease play a crucial role in preventing the spread of these diseases, because it gives a vision of how these diseases spread and helps public authorities develop plans to reduce their spread or block it. The simplest structure of epidemic mathematical model is the SIS model formulated by Kermack and McKendrick [1] in which the total population (N) is divided into two classes, susceptible class (S) and infected class (I). The deterministic SIS model with mass action incidence is given by

$$\begin{cases} dS(t) = & [A - \mu S(t) - \beta S(t)I(t) + \lambda I(t)]dt, \\ dI(t) = & [\beta S(t)I(t) - (\mu + \lambda + \alpha)I(t)]dt. \end{cases} \tag{1.1}$$

The parameters in the model 1.1 are summarized in the following

A: a constant input of new members into the population.

β: the disease transmission coefficient between compartments S and I.

μ: the natural death rate of Susceptible and infected individuals.

λ: the recovery rate of infected individuals. α: the disease caused death rate of infectious individuals.

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As is well known, the environment is full of randomness and stochasticity. So, a lot of researchers studied the effect of randomness on the spread of infectious diseases [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]. To include stochasticity into a deterministic model, there are many approches. We mention some approaches that are adopted most often. The first one is with parameters perturbation [2, 3, 4, 5, 6]. The second one is through time Markov chain [7, 8, 9, 10]. The last one is to pertube around the positive equilibria of deterministic models [11, 12, 13].

Several stochastic epidemic models are established by using the first approche mentioned above. Gray and his colleagues [14], formulated and analysed a stochastic version of the deterministic model 1.1 by using a Gaussian white noise to perturb the disease transmission coefficient β. In [15], the authors studied a stochastic SIS epidemic model with media coverage by perturbing the natural death rate μ by a Gaussian white noise.

In this paper, we consider that the rates μ and β are simultaneously subject to random white noises. We replace μdt and βdt in system 1.1 by $\mu dt + \sigma_1 dB_1(t)$ and $\beta dt + \sigma_2 dB_2(t)$ respectively, where $B_i(t)$ is a standard Brownian motion and σ_i the intensity of its corresponding noise, for all $i \in \{1,2\}$. Thus, we get the following new class of the stochastic SIS epidemic model

$$\begin{cases} dS(t) = & [A - \mu S(t) - \beta S(t)I(t) + \lambda I(t)]dt - \sigma_1 S(t)dB_1(t) - \sigma_2 S(t)I(t)dB_2(t), \\ dI(t) = & [\beta S(t)I(t) - (\mu + \lambda + \alpha)I(t)]dt - \sigma_1 I(t)dB_1(t) + \sigma_2 S(t)I(t)dB_2(t). \end{cases} \tag{1.2}$$

We focus here on establishing sufficient conditions for the existence of an ergodic stationary distribution for model (1.2). Ergodicity is an important concept which can help us to predict the behaviour of the model (1.2) for a long time. The ergodicity means that the statistical properties of the stochastic model coincide with the temporal average of its solutions, which implies that the disease will prevail.

The rest of the paper is organized as follows. In section 2, we present some Lemmas an Theorems which will be used in the following sections. In section 3, sufficient conditions are given under which the model 1.2 has a unique stationary distribution. Finally, we close the paper with a brief conclusion.

2. Background material

Let $\mathbb{R}_+^d = \left\{(x_1, \dots, x_d) \in \mathbb{R}^d : x_i > 0, i = 1, \dots, d\right\}$, and $\langle \varphi(t) \rangle = \frac{1}{t}\int_0^t \varphi(r)dr.$

Concerning the existence and uniqueness of the global solution of model 1.2, we present the following Lemma.

Theorem 2.1

For any initial value $(S(0), I(0)) \in \mathbb{R}_+^2$, there is a unique solution $(S(t), I(t)) \in \mathbb{R}_+^2$ and it remains in $\mathbb{R}_+^2$ with probability one.

Since the proof of the lemma is standard [16], we omit it here.

Now , we consider the d-dimensional stochastic differential equation

$$dx(t) = f(x(t),t)dt + g(x(t),t)dB(t) \quad for\ all \quad t \geq t_0, \qquad (2.1)$$

with initial value $x(0) = x_0 \in \mathbb{R}^d$. $B(t)$ denotes an n-dimensional standard Brownian. Denote by $\mathcal{C}^{2,1}(\mathbb{R}^d \times [t_0,\infty]; \mathbb{R}_+)$ the family of all nonnegative functions $V(x,t)$ defined on $\mathbb{R}^d \times [t_0,\infty]$ such that they are continuously twice differentiable in x and once in t. The differential operator L of Equation (2.1) is defined by

$$L = \frac{\partial}{\partial t} + \sum_{i=1}^{d} f_i(x,t)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{d} [g^T(x,t)g(x,t)]_{ij}\frac{\partial^2}{\partial x_i \partial x_j}.$$

Next, we shall present a lemma which gives a criterion for the existence of a stationary distribution to system (1.2).

Let $X(t)$ be a homogeneous Markov process in E_d (E_d denotes $d-$dimensional Euclidien space), and be described by the following stochastic differential equation

$$dX(t) = h(X(t))dt + \sum_{r=1}^{k} g_r(X(t))dB_r(t)$$

The diffusion matrix is defined as follows

$$A(X) = (a_{ij}(x)),\ a_{ij}(X) = \sum_{r=1}^{k} g_r^i(X)g_r^j(X).$$

Lemma 2.3

[18, Chapter 4] The Markov process $\mathrm{X(t)}$ has a unique stationary distribution $\delta(.)$ if there exists a bounded domain $\mathrm{D} \subset \mathrm{E_d}$ with regular boundary Γ and

(A.1) In the open domain U and some neighborhood thereof, the smallest eigenvalue of the diffusion matrix $A(X)$

is bounded away from zero.

(A.2) If there exists a nonnegative $\mathcal{C}^2$-function V such that LV is negative on $E_d \backslash D$.

Then

$$\mathbb{P}_{\mathrm{x}}\left\{\lim_{t\to\infty}\frac{1}{t}\int_0^t \Phi(\mathrm{X(s)})\mathrm{ds} = \int_{\mathrm{E_d}} \Phi(\mathrm{x})\delta(\mathrm{x})\mathrm{dx}\right\} = 1,$$

for all $x \in E_d$, where $\Phi(.)$ is a function integrable with respect to the mesure δ.

3. The stationary distribution of the solution

Let's denote by $R_0 = \frac{\beta A}{\mu(\mu+\lambda+\alpha)}$ the basic reproduction number of the deterministic model (1.1) and by $(S_*, I_*) =$

$(\frac{\mu+\lambda+\alpha}{\beta}, \frac{\mu(\mu+\lambda+\alpha)}{\beta(\mu+\alpha)}(R_0-1))$ its endemic equilibrium.

Theorem 3.1

If $R_0 > 1$ *and* $0 < \lambda_3 < min(\lambda_1 S_*^2,\ \lambda_2 I_*^2)$, the model 1.2 has a unique stationary

distribution, where

$\lambda_1 = \mu - \frac{3}{2}\sigma_1^2 - \frac{2\mu+\alpha}{\beta} I_* \sigma_2^2,$

$\lambda_2 = \mu + \alpha - \frac{3}{2}\sigma_1^2,$

$\lambda_3 = 3\sigma_1^2(S_*^2 + I_*^2) + \frac{(2\mu+\alpha)I_*}{\beta}(\frac{\sigma_1^2}{2} + S_*^2\sigma_2^2).$

Proof. To show that the model (1.2) has a unique stationary distribution, we only need to validate conditions **(A.1)** and **(A.2)** in Lemma 2.3.

Now we verify the condition **(A.1)**.

Let D be the full ellipse defined by

$D = \{(S(t), I(t)) \in \mathbb{R}_+^2 : \lambda_1(S(t) - S_*)^2 + \lambda_2(I(t) - I_*)^2 \le \lambda_3\}.$

The diffusion matrix of the model 1.2 is

$$\Delta(S(t), I(t)) = \begin{pmatrix} \sigma_1^2 S^2(t) + \sigma_2^2 S^2(t) I^2(t) & \sigma_1^2 S(t) I(t) - \sigma_2^2 S^2(t) I^2(t) \\ \sigma_1^2 S(t) I(t) - \sigma_2^2 S^2(t) I^2(t) & \sigma_1^2 I^2(t) + \sigma_2^2 S^2(t) I^2(t) \end{pmatrix}.$$

Let $\big(S(t), I(t)\big) \in \mathbb{R}_+^2$ and $(\xi_1,\ \xi_2) \in \mathbb{R}_+^2$. Then

$\sum_{i,j=1}^2 \Delta_{ij}\big(S(t), I(t)\big)\xi_i \xi_j \ge \eta(\xi_1^2 + \xi_2^2),$

Where

$$\eta = \min_{(S(t),I(t))\in D}(\sigma_1^2 S^2(t), \sigma_1^2 I^2(t), \sigma_2^2 S^2(t) I^2(t))$$

Thus, condition **(A.1)** holds.

To show that the condition **(A.2)** is verified, we define a C^2-function $V\colon\ \mathbb{R}_+^2 \longrightarrow \mathbb{R}_+$ as follows

$V(S, I) = \frac{2\mu+\alpha}{\beta}[I - I_* - I_* \ln(\frac{I}{I_*})] + \frac{1}{2}(S - S_* + I - I_*)^2 = \frac{2\mu+\alpha}{\beta} V_1 + V_2.$

Making use of the Itô's formula [17], we obtain

$$LV_1 = (I - I_*)[\beta S - (\mu + \lambda + \alpha)] + \frac{\sigma_1^2}{2} I_* + \frac{\sigma_2^2}{2} I_* S^2,$$

$$\leq (I - I_*)[\beta S - \beta S_*] + \frac{\sigma_1^2}{2} I_* + \frac{\sigma_2^2}{2} I_*[(S - S_*) + S_*]^2,$$

$$\leq \beta(S - S_*)(I - I_*) + \frac{\sigma_1^2}{2} I_* + I_* \sigma_2^2 (S - S_*)^2 + I_* \sigma_2^2 S_*^2.$$

On the other hand, we have

$$LV_2 = (S - S_* + I - I_*)(A - \mu S - (\mu + \alpha)I) + \frac{\sigma_1^2}{2}(S + I)^2,$$

$$= (S - S_* + I - I_*)\big(-\mu(S - S_*) - (\mu + \alpha)(I - I_*)\big)$$

$$+ \frac{\sigma_1^2}{2}[(S - S_*) + \quad (I - I_*) + (S_* + I_*)\,]^2,$$

$$\leq -\mu(S - S_*)^2 - (\mu + \alpha)(I - I_*)^2 - (2\mu + \alpha)(S - S_*)(I - I_*)$$

$$+ \frac{3}{2}\sigma_1^2(S - S_*)^2 + \frac{3}{2}\sigma_1^2(I - I_*)^2 + \frac{3}{2}\sigma_1^2(S_* + I_*)^2,$$

$$= -(\mu - \frac{3}{2}\sigma_1^2)(S - S_*)^2 - (\mu + \alpha - \frac{3}{2}\sigma_1^2)(I - I_*)^2$$

$$-(2\mu + \alpha)(S - S_*)(I - I_*) + 3\sigma_1^2 S_*^2 + 3\sigma_1^2 I_*^2.$$

Hence

$$LV \leq -[\mu - \frac{3}{2}\sigma_1^2 - \frac{2\mu+\alpha}{\beta} I_* \sigma_2^2](S - S_*)^2 - [\mu + \alpha - \frac{3}{2}\sigma_1^2](I - I_*)^2$$

$$+[3\sigma_1^2(S_*^2 + I_*^2) + \frac{(2\mu+\alpha)I_*}{\beta}(\frac{\sigma_1^2}{2} + S_*^2\sigma_2^2)],$$

$$= -[\lambda_1(S - S_*)^2 + \lambda_2(I - I_*)^2 - \lambda_3].$$

Finally

$$LV(S, I) < 0 \qquad for\ \ all \quad (S, I) \in \mathbb{R}_+^2 \backslash D,$$

which means that **(A.2)** holds.

According to Lemma 2.3, we conclude that system (1.2) is ergodic and admits a unique stationary distribution.

5. Conclusion

In this paper, we investigated a class of stochastic SIS epidemic model with a Complex type of noises. The only stochastic component is due to the environmental variability, which we incoporate in the model as white noise about the disease transmission rate and the natural death rate. We showed that the solution of the model 1.2 has a unique stationary distribution when the intensity of noise is small. This distribution has the ergodic property, which means that the statistical behavior of the model can be known by the temporal average of the solution . Besides, under the stationarity condition, the disease will prevail.

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